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Sharper global existence for the generalized 1D nonhomogeneous Ginzburg–Landau equation ☆

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Abstract

We study the following generalized 1D Ginzburg–Landau equation on $\Omega = (0,\infty) \times (0,\infty)$:

$$u_t = (1 + i\mu)u_{xx} + (a_1 + ib_1)|u|^2 u_x + (a_2 + ib_2)u^2 \bar{u}_x - (1 + i\nu)|u|^4 u$$

with initial and Dirichlet boundary conditions $u(x,0) = h(x)$, $u(0,t) = Q(t)$. Based on detail analysis, the sharper existence and uniqueness of global solutions are obtained under sufficient conditions.

Keywords: Generalized 1D nonhomogeneous Ginzburg–Landau equation; Global existence

1. Introduction

The cubic Ginzburg–Landau equation (GL)

$$u_t = (1 + i\mu)u_{xx} - (1 + i\nu)|u|^2 u + \nu u \tag{1.1}$$

is often used to describe the amplitude evolution of instability waves in a large variety of dissipative systems, for example, in fluid dynamics, chemical reaction, superconductivity, etc. It frequently occurs as the leading term in an asymptotic expansions of envelope solutions for exact models such as Navier–Stokes equation [1]. It is of physical interest to carry the expansion to second order [2,3]. It leads to the following generalized GL:

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$$u_t = (1 + i\mu)u_{xx} + (a_1 + ib_1)|u|^2 u_x + (a_2 + ib_2)u^2 \bar{u}_x - (1 + i\nu)|u|^4 u - (a_0 + ib_0)|u|^2 u + \gamma u. \tag{1.2}$$

If $4 > (b_1 - b_2)^2$ then (1.2) with initial data $u(x, 0) \in H^1_{\text{per}}[0, L]$ possesses a global solution in $C([0, \infty); H^1_{\text{per}}[0, L]) \cap C^1([0, \infty); H^1_{\text{per}}[0, L])$ (see [5]). It also has been found that the cubic terms involving partial derivatives can significantly slow the propagation speed of moving fronts and pulses [2] and must be balanced by the fifth order term and the second derivative term.

There are many papers in the literature regarding the GL equation and its generalized versions in one and two dimension [4–14]. However, most of them study the initial value or periodic boundary value problems. Yet, in many cases of physical interest, the mathematical models lead to the forced problems when a nonzero boundary condition is imposed [15,16]. Forced problems occur when an external force is applied to the time evolution of systems governed by nonlinear partial differential equation. The forcing is often put in as a boundary condition of Dirichlet or Neumann type when $x = 0$. To establish the model for the forced GL equation, one considers the unstable waves specified for all times at the reflection point, and asks for the time development if the nonlinear phenomena at that point. The relevant mathematical model reads as

$$u_t = (1 + i\mu)u_{xx} - (1 + i\nu)|u|^2 u + \gamma u, \tag{1.3}$$

$$u(x, 0) = h(x), \qquad u(0, t) = Q(t). \tag{1.4}$$

For (1.3)–(1.4), a global classical solution is available when $\mu\nu > 0$ or $|\nu| \leqslant \sqrt{3}$ [17]. For the GL equation posed in the finite domain $0 \leqslant x \leqslant L$, $0 \leqslant t < \infty$ with Dirichlet or Neumann boundary condition, a unique weak solution in H^1 can be obtained by using the Galerkin–Vishik method [18,19].

For a generalized GL equation (1.2) with initial condition and nonzero Dirichlet condition, in [20] Gao and Bu obtained the global existence and uniqueness when $|\nu| < \sqrt{5}/2$ and $(|a_1| + |a_2| + |b_1| + |b_2|)^2 < 2(3 - 2\sqrt{1 + \nu^2})$.

In this paper, we will give a sharper conditions than [20] to guarantee the global existence of (1.2) with nonzero Dirichlet boundary condition.

2. Global existence theorem

We study the following initial-boundary value problem for a generalized Ginzburg–Landau equation:

$$u_t = (1 + i\mu)u_{xx} + (a_1 + ib_1)|u|^2 u_x + (a_2 + ib_2)u^2 \bar{u}_x - (1 + i\nu)|u|^4 u, \tag{2.1}$$

$$u(x, 0) = h(x), \quad x \geqslant 0, \tag{2.2}$$

$$u(0, t) = Q(t), \quad t \geqslant 0. \tag{2.3}$$

Throughout this paper, $\|\cdot\|$ and $\|\cdot\|_p$ denote the norm of L^2 and L^p, respectively, and $u_x(0, t) = P$.

The main result of this paper is

Theorem 2.1. *Let* $h \in H^1 = H^1(R^+)$, $Q \in C^1([0,\infty))$, $Q(0) = h(0)$. *For the solution* $u(t)$ *of problem* (2.1)–(2.3), *with one of the following conditions are held*:

(1) $|\mu| < \sqrt{5}/2$, ν *arbitrary*, $M^2 < 3 - 2\sqrt{1+\mu^2}$,
(2) $|\mu| = \sqrt{5}/2$, ν *arbitrary*, $M^2 < 3 - 2\sqrt{1+\alpha^2}$, $|\alpha| < \sqrt{5}/2$,
(3) $\mu\nu > 0$, $M^2 < 1$,
(4) $\mu\nu < 0$, $|\nu| < \sqrt{5}/2$, $|\mu| > \sqrt{5}/2$, $M^2 < 3 - 2\sqrt{1+\nu^2}$,
(5) $\mu\nu < 0$, $|\nu| > \sqrt{5}/2$, $|\mu| > \sqrt{5}/2$, $|\alpha| < \sqrt{5}/2$, $-(1+\mu\nu) < |\alpha||\mu-\nu|$, $M^2 < 3 - 2\sqrt{1+\alpha^2}$,

we have

$$\|u\|_{H^1} \leqslant K \quad \text{for } 0 \leqslant t \leqslant T, \text{ for any given } T > 0,$$

where $M = \max\{|a_1 + ib_1|, |a_2 + ib_2|\}$, K *is a constant which depends on initial value, boundary data, parameters in Eq.* (2.1) *and* T. *For the local existence of solution* (*we can refer to* [20]), *then by the local existence and the above estimate, there is a unique global weak solution* $u \in C([0,T); H^1) \cap C^1((0,T); H^1)$ *of* (2.1)–(2.3) *for any given* $T > 0$.

In order to get the proof of above theorem we need the following lemma.

Lemma 2.2. *Let* u *be a smooth solution to the initial-boundary value problem for the Ginzburg–Landau equation* (2.1)–(2.3). *Then the following identities are available. First,*

$$\frac{d}{dt}\|u\|^2\,dx = -2\,\mathrm{Re}(1+i\mu)P\bar{Q} - 2\|u_x\|^2 - \frac{1}{2}(a_1+a_2)|Q|^4 + 2(b_1-b_2)\,\mathrm{Im}\int_0^{+\infty}|u|^2u\bar{u}_x\,dx - 2\int_0^{+\infty}|u|^6\,dx. \tag{2.4}$$

Second,

$$\begin{aligned}\frac{d}{dt}\int_0^{+\infty}u\bar{u}_x\,dx\,dt &= \int_0^{+\infty}(u\bar{u}_{xt}+u_t\bar{u}_x)\,dx = -Q\bar{Q}' + 2i\,\mathrm{Im}\int_0^{+\infty}\bar{u}_xu_t\,dx\\ &= -Q\bar{Q}' - 2i\mu|P|^2 - 2i\,\mathrm{Im}\int_0^{+\infty}(1+i\mu)\bar{u}_{xx}u_x\,dx + 2i\int_0^{+\infty}b_1|u|^2|u_x|^2\,dx\\ &\quad + 2i\,\mathrm{Im}\int_0^{+\infty}(a_2+ib_2)u^2\bar{u}_x^2\,dx - 2i\,\mathrm{Im}\int_0^{+\infty}(1+i\nu)|u|^4u\bar{u}_x\,dx.\end{aligned} \tag{2.5}$$

Let

$$E_\delta(t) = \int_0^{+\infty}\left(\frac{1}{2}|u_x|^2 + \frac{\delta}{6}|u|^6\right)dx, \quad \delta \text{ to be chosen later}.$$

Then

$$
\begin{aligned}
\frac{d}{dt}E_\delta(t) &= -\operatorname{Re}\bar{P}Q' - \operatorname{Re}\int_0^{+\infty} u_t\bar{u}_{xx}\,dx + \delta\operatorname{Re}\int_0^{+\infty}|u|^4\bar{u}u_t\,dx \\
&= -\operatorname{Re}\bar{P}Q' - \operatorname{Re}\int_0^{+\infty}\big[(1+i\mu)|u_{xx}|^2 + (a_1+ib_1)|u|^2u_x\bar{u}_{xx} \\
&\qquad + (a_2+ib_2)u^2\bar{u}_x\bar{u}_{xx} - (1+i\nu)|u|^4u\bar{u}_{xx}\big]\,dx \\
&\quad + \delta\operatorname{Re}\int_0^{+\infty}\big[(1+i\mu)u_{xx}|u|^4\bar{u} + (a_1+ib_1)|u|^6\bar{u}u_x \\
&\qquad + (a_2+ib_2)u^2\bar{u}_x|u|^4\bar{u} - (1+i\nu)|u|^{10}\big]\,dx. \qquad (2.6)
\end{aligned}
$$

Proof. We first consider the case of $\mu \neq 0$. (2.4) can be easily verified by integrating by parts, substituting the original equation, and using

$$
\operatorname{Re}\int_0^{+\infty}|u|^2u\bar{u}_x\,dx = \int_0^{+\infty}|u|^2\operatorname{Re}(u\bar{u}_x)\,dx = \frac{1}{2}\int_0^{+\infty}|u|^2d|u|^2\,dx = -\frac{1}{4}|Q|^4.
$$

Here, $u(+\infty,t)=0$, $t \geqslant 0$, is used since $u(t) \in H^1(R^+)$. Equations (2.5) and (2.6) can be obtained by integrating by parts and using the original equation. The proof of Lemma 2.2 is completed. □

Proof of Theorem 2.1. By Cauchy inequality, (2.4) becomes

$$
\begin{aligned}
\frac{d}{dt}\|u\|^2 &\leqslant -2\operatorname{Re}(1+i\mu)P\bar{Q} - 2\|u_x\|^2 - \frac{1}{2}(a_1+a_2)|Q|^4 \\
&\quad + \frac{|b_1-b_2|^2}{2}\int_0^{+\infty}|u_x|^2\,dx + 2\|u\|_6^6 - 2\int_0^{+\infty}|u|^6\,dx \\
&\leqslant c_1|P| - 2\|u_x\|^2 - \frac{1}{2}(a_1+a_2)|Q|^4 + \frac{|b_1-b_2|^2}{2}\int_0^{+\infty}|u_x|^2\,dx \\
&\leqslant m|P|^2 + c_2 - 2\|u_x\|^2 + \frac{|b_1-b_2|^2}{2}\int_0^{+\infty}|u_x|^2\,dx \\
&= m|P|^2 + c_2 + c_0\|u_x\|^2, \qquad (2.7)
\end{aligned}
$$

where

$$
c_0 = \frac{|b_1-b_2|^2}{2} - 2, \qquad c_1 = 2|1+i\mu|\max_{0\leqslant t<\infty}\big|Q(t)\big|,
$$

and $m > 0$ to be determined later, and

$$c_2 = \frac{{c_1}^2}{4m} - \frac{1}{2}(a_1 + a_2) \max_{0 \leqslant t < \infty} \left|Q(t)\right|^4.$$

By integrating in t variable on both sides of (2.7), we obtain

$$\|u\|^2 \leqslant m \int_0^t |P|^2\, dt + c_3 + c_0 \int_0^t \|u_x\|^2\, dt \tag{2.8}$$

for some $c_3 = \|h\|^2 + c_2 t$.

By taking the imaginary part on both sides of (2.5) and integrating in t variable, we obtain

$$\begin{aligned} 2\mu \int_0^t |P|^2\, dt = {} & -\operatorname{Im} \int_0^{+\infty} u\bar{u}_x\, dx + \operatorname{Im} \int_0^{+\infty} h\bar{h}'\, dx - \operatorname{Im} \int_0^t Q\bar{Q}'\, dt \\ & - 2\operatorname{Im} \int_0^t \int_0^{+\infty} (1 + i\mu)\bar{u}_{xx} u_x\, dx\, dt \\ & - 2\operatorname{Im} \int_0^t \int_0^{+\infty} (1 + i\nu)|u|^4 u\bar{u}_x\, dx\, dt \\ & + 2 \int_0^t \int_0^{+\infty} b_1 |u|^2 |u_x|^2\, dx\, dt + 2\operatorname{Im} \int_0^t \int_0^{+\infty} (a_2 + ib_2) u^2 \bar{u}_x^2\, dx\, dt. \end{aligned} \tag{2.9}$$

Therefore,

$$\begin{aligned} \int_0^t |P|^2\, dt \leqslant {} & \frac{1}{|\mu|}\left(\|u\|^2 + \|u_x\|^2\right) + c_4 + \mu_0 \int_0^t \left(\|u_x\|^2 + \|u_{xx}\|^2\right) dt \\ & + c_5 \int_0^t \int_0^{+\infty} \left(|u|^6 + |u|^4 |u_x|^2 + |u_x|^2\right) dx\, dt \end{aligned} \tag{2.10}$$

for some $\mu_0 = \sqrt{1 + |\mu|^2}/|\mu|$ and some $c_4 > 0$ depends only on $\|h\|_{H^1}$, $\|Q\|_{C^1[0,\infty)}$, and $|\mu|$; $c_5 > 0$ depends on a_1, a_2, b_1, b_2, $|\mu|$, and $|\nu|$. Now we substitute (2.8) in (2.10) to get

$$\begin{aligned} \int_0^t |P|^2\, dt \leqslant {} & \frac{1}{|\mu|}\left(m \int_0^t |P|^2\, dt + c_3 + c_0 \int_0^t \|u_x\|^2\, dt + \|u_x\|^2\right) + c_4 \\ & + \mu_0 \int_0^t \left(\|u_{xx}\|^2 + \|u_x\|^2\right) dt \end{aligned}$$

$$+ c_5 \int_0^t \int_0^{+\infty} \left(|u|^6 + |u|^4|u_x|^2 + |u_x|^2\right) dx\, dt,$$

$$\begin{aligned}\left(1 - \frac{m}{|\mu|}\right) \int_0^t |P|^2\, dt \leqslant{}& \frac{c_3}{|\mu|} + \frac{c_0}{|\mu|} \int_0^t \|u_x\|^2\, dt + \frac{1}{|\mu|}\|u_x\|^2 + c_4 \\ &+ \mu_0 \int_0^t \left(\|u_{xx}\|^2 + \|u_x\|^2\right) dt \\ &+ c_5 \int_0^t \int_0^{+\infty} \left(|u|^6 + |u|^4|u_x|^2 + |u_x|^2\right) dx\, dt. \qquad (2.11)\end{aligned}$$

Here, we chose m such that $m < |\mu|$. Multiplying $m|\mu|/(|\mu| - m)$ on both sides of (2.11), it can be written as

$$\begin{aligned} m\int_0^t |P|^2\, dt \leqslant{}& c_6 + \frac{mc_0}{|\mu| - m} \int_0^t \|u_x\|^2\, dt + \frac{m}{|\mu| - m}\|u_x\|^2 \\ &+ \frac{m\mu_0|\mu|}{|\mu| - m} \int_0^t \left(\|u_{xx}\|^2 + \|u_x\|^2\right) dt \\ &+ \frac{c_5 m|\mu|}{|\mu| - m} \int_0^t \int_0^{+\infty} \left(|u|^6 + |u|^4|u_x|^2 + |u_x|^2\right) dx\, dt, \qquad (2.12)\end{aligned}$$

where

$$c_6 = \frac{|\mu|m}{|\mu| - m}\left(\frac{c_3}{|\mu|} + c_4\right).$$

By (2.4) and (2.6), we have

$$\begin{aligned} \frac{d}{dt}\left[E_\delta(t) + \|u\|^2\right] ={}& \operatorname{Re} \int_0^{+\infty} \left(\bar{u}_x u_{xt} + \delta|u|^4 \bar{u} u_t\right) dx + \frac{d}{dt}\|u\|^2 \\ ={}& -\operatorname{Re} \bar{P} Q' - \operatorname{Re} \int_0^{+\infty} \big[(1 + i\mu)|u_{xx}|^2 + (a_1 + ib_1)|u|^2 u_x \bar{u}_{xx} \\ &+ (a_2 + ib_2)u^2 \bar{u}_x \bar{u}_{xx} - (1 + i\nu)|u|^4 u\bar{u}_{xx}\big]\, dx \\ &+ \delta \operatorname{Re} \int_0^{+\infty} \big[(1 + i\mu)u_{xx}|u|^4\bar{u} + (a_1 + ib_1)|u|^6 \bar{u} u_x \\ &+ (a_2 + ib_2)u^2\bar{u}_x|u|^4\bar{u} - (1 + i\nu)|u|^{10}\big]\, dx \end{aligned}$$

$$- 2\,\mathrm{Re}(1+i\mu)P\bar{Q} - 2\|u_x\|^2 - \frac{1}{2}(a_1+a_2)|Q|^4 + 2(b_1-b_2)\,\mathrm{Im}\int_0^{+\infty}|u|^2u\bar{u}_x\,dx - 2\int_0^{+\infty}|u|^6\,dx$$

$$\leqslant -\mathrm{Re}\,\bar{P}Q' - \|u_{xx}\|^2 + 2M\int_0^{+\infty}|u|^2|u_xu_{xx}|\,dx + \mathrm{Re}(1+i\nu)\int_0^{+\infty}|u|^4u\bar{u}_{xx}\,dx$$

$$+ \delta\,\mathrm{Re}(1+i\mu)\int_0^{+\infty}|u|^4\bar{u}u_{xx}\,dx + 2M\delta\int_0^{+\infty}|u|^7|u_x|\,dx - \delta\|u\|_{10}^{10}$$

$$- 2\,\mathrm{Re}(1+i\mu)P\bar{Q} - \frac{1}{2}(a_1+a_2)|Q|^4 + c_0\|u_x\|^2$$

$$\leqslant -\mathrm{Re}\,\bar{P}Q' - \big(\|u_{xx}\|^2 + \delta\|u\|_{10}^{10}\big) + \frac{1}{2}\,\mathrm{Re}\int_0^{+\infty}\big(|u|^4u, u_{xx}\big)N_0\big(|u|^4\bar{u}, \bar{u}_{xx}\big)^t\,dx$$

$$+ 2M\int_0^{+\infty}|u|^2|u_xu_{xx}|\,dx + 2M\delta\int_0^{+\infty}|u|^7|u_x|\,dx$$

$$- 2\,\mathrm{Re}(1+i\mu)P\bar{Q} - \frac{1}{2}(a_1+a_2)|Q|^4 + c_0\|u_x\|^2, \tag{2.13}$$

where $(|u|^4\bar{u}, \bar{u}_{xx})^t$ denotes the transpose of $(|u|^4\bar{u}, \bar{u}_{xx})$ and

$$N_0 = \begin{pmatrix} 0 & 1+\delta-i(\mu\delta-\nu) \\ 1+\delta+i(\mu\delta-\nu) & 0 \end{pmatrix}.$$

Since for any α satisfies $|\alpha| < \sqrt{5}/2$ (see [21,22]), we have

$$-\mathrm{Re}\int_0^{+\infty}(1+i\alpha)|u|^4\bar{u}u_{xx}\,dx$$

$$\geqslant \big(3 - 2\sqrt{1+\alpha^2}\big)\int_0^{+\infty}|u|^4|u_x|^2\,dx + \mathrm{Re}(1+i\alpha)|Q|^4\bar{Q}P. \tag{2.14}$$

Multiplying (2.14) by $-\eta$ ($\eta > 0$ to be chosen) and adding it to (2.13), since

$$2M\int_0^{+\infty}|u|^2|u_xu_{xx}|\,dx \leqslant \frac{M^2}{1-k-\epsilon}\int_0^{+\infty}|u|^4|u_x|^2\,dx + (1-k-\epsilon)\int_0^{+\infty}|u_{xx}|^2\,dx,$$

$$2\delta M\int_0^{+\infty}|u|^7|u_x|\,dx \leqslant (1-k-\epsilon)\delta\|u\|_{10}^{10} + \frac{M^2\delta}{1-k-\epsilon}\int_0^{+\infty}|u|^4|u_x|^2\,dx,$$

where $0<\epsilon<1-k$, $k>0$. We obtain

$$
\begin{aligned}
&\frac{d}{dt}\bigl[E_\delta(t)+\|u\|^2\bigr]\\
&\leqslant -\operatorname{Re}\bar{P}Q'-(1-k)\bigl(\|u_{xx}\|^2+\delta\|u\|_{10}^{10}\bigr)-\eta\bigl(3-2\sqrt{1+\alpha^2}\bigr)\int_0^{+\infty}|u|^4|u_x|^2\,dx\\
&\quad-\operatorname{Re}(1+i\alpha)\eta|Q|^4\bar{Q}P+\frac{1}{2}\operatorname{Re}\int_0^{+\infty}\bigl(|u|^4u,u_{xx}\bigr)N\bigl(|u|^4\bar{u},\bar{u}_{xx}\bigr)^t\,dx\\
&\quad+\frac{M^2(1+\delta)}{1-k-\epsilon}\int_0^{+\infty}|u|^4|u_x|^2\,dx+(1-k-\epsilon)\|u_{xx}\|^2+(1-k-\epsilon)\delta\|u\|_{10}^{10}\\
&\quad-2\operatorname{Re}(1+i\mu)P\bar{Q}-\frac{1}{2}(a_1+a_2)|Q|^4+c_0\|u_x\|^2\\
&\leqslant c|P|-\eta\bigl(3-2\sqrt{1+\alpha^2}\bigr)\int_0^{+\infty}|u|^4|u_x|^2\,dx\\
&\quad+\frac{1}{2}\operatorname{Re}\int_0^{+\infty}\bigl(|u|^4u,u_{xx}\bigr)N\bigl(|u|^4\bar{u},\bar{u}_{xx}\bigr)^t\,dx\\
&\quad+\frac{M^2(1+\delta)}{1-k-\epsilon}\int_0^{+\infty}|u|^4|u_x|^2\,dx-\epsilon\|u_{xx}\|^2\\
&\quad-\frac{1}{2}(a_1+a_2)|Q|^4+c_0\|u_x\|^2,
\end{aligned}
\tag{2.15}
$$

where

$$
N=\begin{pmatrix}-2\delta k & 1+\delta-\eta-i(\mu\delta-\nu-\alpha\eta)\\ 1+\delta-\eta+i(\mu\delta-\nu-\alpha\eta) & -2k\end{pmatrix}.
$$

By integrating in t variable on both sides of (2.15), we have

$$
\begin{aligned}
&E_\delta(t)+\|u\|^2\\
&\leqslant m\int_0^t|P|^2\,dt+c_7+\left[\frac{M^2(1+\delta)}{1-k-\epsilon}-\eta\bigl(3-2\sqrt{1+\alpha^2}\bigr)\right]\int_0^t\int_0^{+\infty}|u|^4|u_x|^2\,dx\,dt\\
&\quad-\epsilon\int_0^t\|u_{xx}\|^2\,dt+\frac{1}{2}\operatorname{Re}\int_0^t\int_0^{+\infty}\bigl(|u|^4u,u_{xx}\bigr)N\bigl(|u|^4\bar{u},\bar{u}_{xx}\bigr)^t\,dx\,dt+c_0\int_0^t\|u_x\|^2
\end{aligned}
\tag{2.16}
$$

for some

$$c_7 = \frac{tc^2}{4m} - \frac{1}{2}(a_1 + a_2)|Q|^4 t + E_\delta(0) + \|h\|^2.$$

Now we substitute (2.12) in (2.16) to get

$$\begin{aligned}
&E_\delta(t) + \|u\|^2 \\
&\quad \leqslant c_6 + \frac{mc_0}{|\mu| - m}\int_0^t \|u_x\|^2\,dt + \frac{m}{|\mu| - m}\|u_x\|^2 + \frac{m|\mu|\mu_0}{|\mu| - m}\int_0^t \big(\|u_{xx}\|^2 + \|u_x\|^2\big)\,dt \\
&\qquad + \frac{c_5 m|\mu|}{|\mu| - m}\int_0^t \int_0^{+\infty} \big(|u|^6 + |u|^4|u_x|^2 + |u_x|^2\big)\,dx\,dt + c_7 \\
&\qquad + \left[\frac{M^2(1+\delta)}{1-k-\epsilon} - \eta\big(3 - 2\sqrt{1+\alpha^2}\big)\right]\int_0^t \int_0^{+\infty} |u|^4|u_x|^2\,dx\,dt \\
&\qquad - \epsilon\int_0^t \|u_{xx}\|^2\,dt + \frac{1}{2}\,\mathrm{Re}\int_0^t \int_0^{+\infty} \big(|u|^4 u, u_{xx}\big) N \big(|u|^4\bar{u}, \bar{u}_{xx}\big)^t\,dx\,dt \\
&\qquad + c_0\int_0^t \|u_x\|^2\,dt.
\end{aligned} \tag{2.17}$$

Choosing m small enough such that

$$\frac{m}{|\mu|} < 1, \qquad \frac{m}{|\mu| - m} \leqslant \frac{1}{2}, \qquad \frac{m\mu_0|\mu|}{|\mu| - m} < 1 - k,$$

then taking $\epsilon = m\mu_0|\mu|/(|\mu| - m)$ and if N is negative semidefinite, we have

$$\begin{aligned}
&E_\delta + \|u\|^2 \\
&\quad \leqslant M_0 + M_1\int_0^t \big(E_\delta + \|u\|^2\big)\,dt \\
&\qquad + \left[\frac{M^2(1+\delta)}{1-k-\epsilon} - \eta\big(3 - 2\sqrt{1+\alpha^2}\big) + \frac{m|\mu|\mu_0}{|\mu| - m}\right]\int_0^t \int_0^{+\infty} |u|^4|u_x|^2\,dx\,dt.
\end{aligned}$$

Here

$$M_0 = 2(C_6 + c_7), \qquad M_1 = 2\left(\frac{mc_0}{|\mu| - m} + \frac{m|\mu|\mu_0}{|\mu| - m} + \frac{c_5 m|\mu|}{|\mu| - m}\right).$$

It is clear that N is negative semidefinite if and only if

$$(1 + \delta - \eta)^2 + (\delta\mu - \nu - \alpha\eta)^2 \leqslant 4\delta k^2. \tag{2.18}$$

Case (a) When $|\mu| < \sqrt{5}/2$, it suffices to take $\eta = 1 + \delta$, $\alpha = \mu$, $0 < k < 1$, and δ large enough, then (2.18) is satisfied.

Case (b) When $|\mu| = \sqrt{5}/2$, ν arbitrary, it suffices to take $\eta = 1+\delta$, $\alpha = -\mu/(1+\delta) + \delta\mu/(1+\delta)$, $0 < k < 1$, and δ large enough, then (2.18) is satisfied.

Case (c) When $\mu\nu > 0$, it suffices to take $\eta = 1 + \delta$, $\alpha = 0$, $0 < k < 1$, and $\delta = \nu/\mu$, then (2.18) is satisfied.

Case (d) When $\mu\nu < 0$, $|\nu| < \sqrt{5}/2$, $|\mu| > \sqrt{5}/2$, it suffices to take $\eta = 1+\delta$, $\alpha = -\nu$, $0 < k < 1$, and δ small enough, then (2.18) is satisfied.

Case (e) When $\mu\nu < 0$, $|\nu| \geqslant \sqrt{5}/2$, $|\mu| > \sqrt{5}/2$, $\alpha < \sqrt{5}/2$, we may restrict to the case $\mu > 0 \geqslant \nu$, $\alpha > 0$, as $\mu < 0 < \nu$, $\alpha < 0$ is similar by conjugation. So we consider the case of $\mu > \sqrt{5}/2$, $\nu \leqslant -\sqrt{5}/2$, and $\alpha > 0$, we take $\eta = 1 + \delta$ and α to satisfy that $-(1 + \mu\nu) < |\alpha||\mu - \nu|$, then (2.18) is satisfied.

By the following conditions (1)–(5) and choosing m small enough, we have

$$\left[\frac{M^2(1+\delta)}{1-k-\epsilon} - \eta\big(3 - 2\sqrt{1+\alpha^2}\big) + \frac{c_5 m|\mu|}{|\mu| - m}\right]\int_0^t\int_0^{+\infty} |u|^4|u_x|^2\,dx\,dt \leqslant 0. \tag{2.19}$$

(1) $|\mu| < \sqrt{5}/2$, ν arbitrary, $M^2 < 3 - 2\sqrt{1+\mu^2}$;
(2) $|\mu| = \sqrt{5}/2$, ν arbitrary, $|\alpha| < \sqrt{5}/2$, $M^2 < 3 - 2\sqrt{1+\alpha^2}$;
(3) $\mu\nu > 0$, $M^2 < 1$;
(4) $\mu\nu < 0$, $|\nu| < \sqrt{5}/2$, $|\mu| > \sqrt{5}/2$, $M^2 < 3 - 2\sqrt{1+\nu^2}$;
(5) $\mu\nu < 0$, $|\nu| \geqslant \sqrt{5}/2$, $\mu > \sqrt{5}/2$, $|\alpha| < \sqrt{5}/2$, $-(1+\mu\nu) \leqslant |\alpha||\mu - \nu|$, $M^2 < 3 - 2\sqrt{1+\alpha^2}$.

So, we get

$$E_\delta + \|u\|^2 \leqslant M_0 + M_1\int_0^t \big(E_\delta + \|u\|^2\big)\,dt.$$

By Gronwall lemma, we have

$$E_\delta + \|u\|^2 \leqslant C(T), \quad \forall T > 0,\ 0 \leqslant t \leqslant T.$$

By the semigroup theory and the above estimate, the local solution u to (2.1)–(2.3) is global. The proof of Theorem 2.1 is finished. □

Remark 2.3. From the results given in this paper, we could find the conditions which are given in condition (1) (since $|\mu| < \sqrt{5}/2$, so $\sqrt{1+\mu^2}$ has same region as $\sqrt{1+\nu^2}$ when $|\nu| < \sqrt{5}/2$), conditions (3) and (4) are better than the result in [20] (i.e., $|\nu| < \sqrt{5}/2$ and $(|a_1| + |a_2| + |b_1| + |b_2|)^2 < 2(3 - 2\sqrt{1+\nu^2}\,)$). Furthermore, we have conditions (2) and (5).

Remark 2.4. When $\mu = 0$, we can use the transformation

$$v = u - e^{-x}Q(t), \qquad v|_{x=0} = 0,$$

$$
\begin{aligned}
v_t = {} & v_{xx} + (a_1 + ib_1)|v|^2 v_x + (a_2 + ib_2) v^2 \bar{v}_x - (1 + i\nu)|v|^4 v \\
& + G_0(Q, Q', e^{-x}) + G_1(Q, v, \bar{v}, v_x, \bar{v}_x),
\end{aligned}
$$

where

$$
\begin{aligned}
G_0 = {} & e^{-x} Q - e^{-x} Q' - (a_1 + ib_1) e^{-3x} Q|Q|^2 \\
& - (a_2 + ib_2) e^{-3x} Q|Q|^2 - (1 + i\nu) e^{-5x} Q|Q|^4
\end{aligned}
$$

and

$$
\begin{aligned}
G_1 = {} & (a_1 + ib_1)\bigl(e^{-x} \bar{Q} v v_x + e^{-x} Q \bar{v} v_x + e^{-2x}|Q|^2 v_x \\
& - e^{-x} Q|v|^2 - e^{-2x}|Q|^2 v - e^{-2x} Q^2 \bar{v}\bigr) \\
& + (a_2 + ib_2)\bigl(2e^{-x} v \bar{v}_x Q + e^{-2x} Q^2 \bar{v}_x - e^{-x} \bar{Q} v^2 - 2e^{-2x}|Q|^2 v\bigr) \\
& - (1 + i\nu)\bigl(2e^{-x} \bar{Q} v^2 |v|^2 + 3e^{-x} Q|v|^4 + 6e^{-2x}|Q|^2 v|v|^2 \\
& + e^{-2x} \bar{Q}^2 v^3 + 3e^{-3x} \bar{Q}|Q|^2 v^2 + 3e^{-2x} Q^2 \bar{v}|v|^2 + 6e^{-3x} Q|Q|^2|v|^2 \\
& + 3e^{-4x}|Q|^4 v + e^{-3x} Q^3 \bar{v}^2 + 2e^{-4x} Q^2 |Q|^2 \bar{v}\bigr).
\end{aligned}
$$

We have $\|G_0\|_{L^2}$ is bounded and G_1 is local Lipschitz continuous from H^1 to L^2. By the direct estimation, we have $\|v(t)\|^2 + \|v_x(t)\|^2$ is bounded for $t \leqslant T$ and any given T, so the global solution of (2.1)–(2.3) is obtained.

In fact, we can also use the method of homogenization of boundary for the case of $\mu \neq 0$, but our method used here is more concise.

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